Vortex transmutation

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Using group theory arguments and numerical simulations, we demonstrate the possibility of changing the vorticity or topological charge of an individual vortex by means of the action of a system possessing a discrete rotational symmetry of finite order. We establish on theoretical grounds a "transmutation pass rule" determining the conditions for this phenomenon to occur and numerically analize it in the context of two-dimensional optical lattices or, equivalently, in that of Bose-Einstein condensates in periodic potentials.

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Vortices are a physical phenomenon common to all complex waves. Defined by a phase singularity implying the vanishing of the wave amplitude their presence is ubiquitous in physics where examples of vortices can be found in as diverse systems as quantized superfluids and superconductors, Bose-Einstein condensates (BEC's), nonlinear optical structures, or low dimensional condensed matter or particle systems (for a review see [1, 2]). The possibility of changing vortex properties using periodic systems is a natural step based on the known example of the different behavior of electrons with or without the presence of a crystal. Like electrons, properties of vortices in a lattice have been shown to be qualitatively different than in a homogeneous medium. Vortices have been numerically predicted to exist in two-dimensional (2D) arrays of coupled waveguides [3], in 2D periodic dielectric media with Kerr nonlinearities —and, equivalently, in 2D BEC's with periodic potentials— [4, 5] and in photonic crystal fibers with defects [6]. They have been experimentally observed in optically-induced square photonic lattices [7, 8]. In all cases the presence of the periodic medium has a strong influence on vortex features thus opening a door for their external manipulation. In this Letter we will show how manipulation by means of an external system owning discrete rotational symmetry can even affect the most intrinsic feature of a vortex, its vorticity or topological charge, leading to a phenomenon that we refer to as "vortex transmutation".

Angular momentum is conserved in a nonlinear medium with $O(2)$ rotational symmetry in the x-y plane described by a first-order evolution equation of the type $L(|\phi|)\phi = -i\partial\phi/\partial z$ for the complex scalar field ϕ. If we consider a solution with well-defined angular momentum $l \in \mathbb{Z}$ (i.e., an eigenfunction of the angular momentum operator $-i\partial/\partial\theta$: $\phi_l = e^{il\theta} f_l(r)$) at a given axial point z_0, evolution will preserve the value of l for all z. In a system possessing a discrete point-symmetry (described by the $\mathcal{C}_n$ and $\mathcal{C}_{nv}$ groups) angular momentum is no longer conserved. However, in this case one can define another quantity $m \in \mathbb{Z}$, the Bloch or pseudo-angular momentum, which is conserved during propagation [9]. The pseudo-angular momentum m plays then the role of l in a system with discrete rotational symmetry. From the group theory point of view, the angular and pseudo-angular momenta l and m are also the indices of the 2D irreducible representations of $O(2)$ and $\mathcal{C}_n$, respectively [10, 11, 12]. Unlike l, the values of m are limited by the order of the point-symmetry group $\mathcal{C}_n$: $|m| \leq n/2$ [9, 11].

The appearance of this upper bound for the pseudo-angular momentum m opens the interesting question of determining the behavior of solutions propagating in a $O(2)$ rotational invariant medium with well-defined angular momentum l after impinging a medium with discrete symmetry of finite order in which the value of l exceeds the upper bound for pseudo-angular momentum. This question can be analyzed in the light of group theory. Let us consider a wave propagating in a $O(2)$ nonlinear medium corresponding to a solution ϕ_l (not necessarily stationary) with well-defined angular momentum l launched into a second nonlinear medium characterized by the $\mathcal{C}_n$ group. The surface separating the two media defines an $O(2)$-$\mathcal{C}_n$ interface that we locate at $z = 0$. We assume evolution is first order in z: $L(|\phi|)\phi = -i\partial\phi/\partial z$. Evolution in the second medium is thus fully determined by the initial condition $\phi_l(0)$. The initial field $\phi_l(0)$ will excite a different representation of the $\mathcal{C}_n$ group depending on the value of l. Once this second wave is excited, it will propagate in the $\mathcal{C}_n$ medium by preserving its representation —defined by its pseudo-angular momentum m. Let φ_m be a function in the representation of $\mathcal{C}_n$ characterized by the index m. Let us determine now what values of l are allowed by symmetry to produce a nonzero projection of $\phi_l(0)$ onto φ_m for a given value of m. The projection coefficient is given by $c_{ml} = \int_{\mathbb{R}^2} \varphi_m^*(r,\theta)\phi_l(r,\theta;0)$. Since φ_m and ϕ_l belong to representations of $\mathcal{C}_n$ and $O(2)$, respectively, they both properly transform under a discrete rotation of order n: $\varphi_m(r,\theta + 2\pi/n) = e^{im(2\pi/n)}\varphi_m(r,\theta)$ and $\phi_l(r,\theta + 2\pi/n) = e^{il(2\pi/n)}\phi_l(r,\theta)$. Thus, by performing the change of variable $\theta \to \theta + 2\pi/n$ in the definition of c_{ml} one arrives to the symmetry relation $c_{ml} = \exp[i(l-m)2\pi/n]c_{ml}$. The c_{ml} coefficient is then

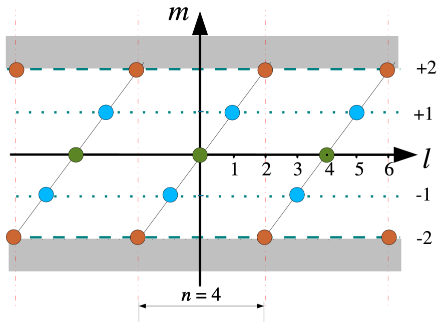

Figure 1: Allowed values of the pseudo-angular momentum m in a $\mathcal{C}_4$ medium in terms of the angular momentum l of the field ϕ_l impinging the interface from an $O(2)$ medium.

zero unless the following condition is fulfilled:

$$l - m = kn \ \ (k \in \mathbb{Z}), \text{ where } \ \ |m| \leq \frac{n}{2}. \qquad (1)$$

The m representation of $\mathcal{C}_n$ is thus excited by initial fields having angular momenta $l = m, m \pm n, m \pm 2n, \ldots$. In a $O(2)$-$O(2)$ interface each representation of angular momentum m in the second medium is excited by one, and only one, angular momentum component l arising from the first medium and verifying $l = m$. Contrarily, in a $O(2)$-$\mathcal{C}_n$ interface the symmetry restriction (1) implies that, due to the cutoff $|m| \leq n/2$ in the $\mathcal{C}_n$ medium, there are infinite angular momenta l that can excite a given representation of index m in the second medium. This can be clearly seen in Fig. 1 in which we represent the permitted values of m for different values of l for the particular case of a second medium with fourth-fold symmetry ($n = 4$) according to the angular momentum "pass rule" in Eq.(1). The effect of the cutoff ($m \leq 2$) is apparent in this representation.

Thus, when the incident field carries an angular momentum l that overcomes the limiting value for pseudo-angular momentum in the second medium it will excite a wave that will propagate with *different* constant pseudo-angular momentum m given by the "pass rule" (1). This result is valid for waves verifying an equation of the type $L(|\phi|)\phi = -i\partial\phi/\partial z$, linear or nonlinear, stationary or evolving. A particularly interesting situation is that in which the incident field is a vortex field of the $O(2)$ nonlinear medium. This vortex field $\phi_l^{\rm v}$ is a stationary solution of the evolution equation with well-defined angular momentum $l \neq 0$: $L(|\phi_l^{\rm v}|)\phi_l^{\rm v} = -\mu\phi_l^{\rm v}$. We consider here individual "canonical" vortices with a single phase singularity (i.e., with only one point in which $\phi_l^{\rm v} = 0$): $\phi_l^{\rm v}(r,\theta,z) = e^{il\theta} f_l^{\rm v}(r) e^{-i\mu z}$ [13]. The vorticity or topological charge of such solutions will be given by the circulation of its phase gradient around the singularity $v = (1/2\pi)\oint \nabla \arg(\phi_l^{\rm v}) d\mathbf{r}$, which equals angular momentum for canonical vortices $v = l$. On the other hand, the propagating wave ϕ_m with pseudo-angular momentum m excited by $\phi_l^{\rm v}$ will evolve in the $\mathcal{C}_n$ medium. There are different options for the asymptotic states of ϕ_m when $z \to \infty$. One possibility is this wave asymptotically tends to an stationary solution $\phi_m \overset{z\to\infty}{\to} \phi_m^{\rm v}(r,\theta,z) = e^{im\theta} g_m(r,\theta) e^{-i\mu' z}$ in the representation of $\mathcal{C}_n$ given by the conserved pseudo-angular momentum m. If $\phi_m^{\rm v}$ has a single phase singularity then it will have the structure of an individual canonical discrete-symmetry vortex, its vorticity or topological charge v' being directly given by m: $v' = m$ [11]. The formation in the $\mathcal{C}_n$ medium of an asymptotic stationary state in the form of a discrete-symmetry vortex is a dynamical issue that depends on the structural parameters of the second medium as well as in the characteristics of the input vortex field $\phi_l^{\rm v}$. If dynamics allows the stabilization of the discrete-symmetry vortex solution, the $O(2)$-$\mathcal{C}_n$ interface will realize the mapping of a $O(2)$ vortex with charge $v = l$ (exceeding the limiting value for pseudo-angular momentum) into a $\mathcal{C}_n$ vortex with charge $v' = m \neq 0$, such that $v' < v$. The "pass rule" for pseudo-angular momentum (1) becomes a "pass rule" relating input and output vorticities:

$$v - v' = kn \ \ (k \in \mathbb{Z}), \qquad (2)$$

where v' presents a cutoff in terms of n given by:$|v'| < n/2$ (even n) and $|v'| \leq (n-1)/2$ (odd n) [11]. Note that $m = n/2$ solutions are not vortices but nodal or dipole-mode solitons [11]. We will refer to the process of mapping an individual vortex into another with different topological charge as "vortex transmutation".

We will provide now a physical example of system in which the phenomenon of "vortex transmutation" takes place. It is an optical interface separating two 2D dielectric media with Kerr nonlinearity, these two media being a homogeneous medium and a 2D square optical lattice. This system is equivalent to a 2D BEC in which a periodic potential is abruptly switched on. They constitute an $O(2)$-$\mathcal{C}_4$ interface given by the following equation:

$$\left(\nabla_t^2 - V(x,y,z) + \gamma(z)|\phi|^2\right)\phi = -i\frac{\partial\phi}{\partial z}, \qquad (3)$$

in which ∇_t is the 2D gradient operator and $V(x,y,z) = V_0 + \theta(z)(V_1(x,y) - V_0)$ where $\theta(z)$ is the step function and V_0 and $V_1(x,y) = V_1\left(\cos^2(\frac{2\pi}{\Lambda}x) + \cos^2(\frac{2\pi}{\Lambda}y)\right)$ (V_1 is the potential strength and Λ is the lattice spatial period) define the refractive index profile of the homogeneous medium and of the 2D optical lattice: $V_0 = -(n^2 - n_0^2)$ and $V_1(\mathbf{x}) = -(n^2(\mathbf{x}) - n_0^2)$, n_0 being a reference refractive index introduced by the slowly-varying envelope approximation. The nonlinear function $\gamma(z) = \gamma + (1-\gamma)\theta(z)$ permits the nonlinear response of the system to be different in the two media. All distances appearing in Eq.(3) are normalized and dimensionless ($\mathbf{x} = k_0\mathbf{x}'$, $z = k_0 z'$). In order to solve the evolution problem in this system we solve first Eq.(3) for

$z < 0$, which becomes an ordinary Nonlinear Schrödinger equation (NLSE) for a homogeneous medium. Since our aim is to evidence the phenomenon of "vortex transmutation" we are interested in finding canonical vortex solitons of different charges in the homogeneous $O(2)$ medium: $\phi_l^{\mathrm{v}}(\mathbf{x}, z) = e^{il\theta} f_l^{\mathrm{v}}(r) e^{-i\mu z}$. This can be done by standard methods. At a given value of l, a family of $O(2)$ vortices are found characterized by their power $P_l = \int_{\mathbb{R}^2} |\phi_l^{\mathrm{v}}|^2$ and their propagation constant μ, which are related through the relation $P_l(\mu)$. In the case of a Kerr nonlinearity, μ behaves as a scaling parameter and P_l is μ-independent [1]. Once the vortex solution ϕ_l^{v} is found, it is taken as an initial solution for propagating it in the 2D optical lattice ($z > 0$): $\phi(\mathbf{x}, 0) = \phi_l^{\mathrm{v}}(\mathbf{x}, 0) = e^{il\theta} f_l^{\mathrm{v}}(r)$. Thus we solve Eq.(3) for $z > 0$, which becomes a NLSE which the periodic potential $V_1(\mathbf{x})$, with the previous initial condition. This is solved numerically using a standard split-step Fourier evolution method.

According to our previous symmetry arguments, the evolution of the ϕ wave for $z > 0$ has to occur in a way that the "pass rule" for angular momentum (1) is fulfilled. The $O(2)$ vortex soliton ϕ_l^{v} carrying angular momentum l will excite a propagating wave ϕ_m for $z > 0$ in a representation of $\mathcal{C}_4$ with pseudo-angular momentum m given by Fig. 1. Indeed, numerical evidence of this "pass rule" is obtained by analyzing the rotational symmetry of the evolving field. By construction, the input momentum is l since we choose the solution to be of the form $\phi_l^{\mathrm{v}}(\mathbf{x}, z) = e^{il\theta} f_l^{\mathrm{v}}(r) e^{-i\mu z}$ for $z \leq 0$. In order to check the symmetry properties of the solution for $z > 0$, we numerically evaluate the rotated field $\bar{\phi}(r, \theta, z) \equiv \phi(r, \theta + \pi/2, z)$ at every step in z and compare it to its unrotated value $\phi(r, \theta, z)$. If ϕ belongs to the m representation of $\mathcal{C}_4$, $\phi(r, \theta + \pi/2, z) = e^{im\pi/2} \phi(r, \theta, z)$ and the ratio $\bar{\phi}/\phi$ will have a constant value for all $\mathbf{x} \in \mathbb{R}^2$ (with the exception of $\mathbf{x} = \mathbf{0}$, where rotations are ill-defined) and $z > 0$: $\bar{\phi}/\phi = e^{im\pi/2}$. If this condition is satisfied the value of m can be directly extracted from the numerical ratio $\bar{\phi}/\phi$. Indeed, the independence of the $\bar{\phi}/\phi$ ratio from transverse coordinates is numerically verified at every axial step, which permits to evaluate m for different values of $z > 0$. Results are shown in Fig. 2. These results nicely confirm the general condition (1) and, more specifically, they satisfy the graphical rule represented in Fig. 1 for an $O(2)$-$\mathcal{C}_4$ interface.

Once the angular momentum "pass rule" is checked there persist the question of the fake of the propagating wave in the $\mathcal{C}_4$ medium. As predicted by theory, m is numerically conserved during evolution. However, the asymptotic behavior of the ϕ_m evolving field can be very different depending on the parameters of the incident vortex field (its power P and its propagation constant μ) and of the characteristics of the periodic potential $V_1(\mathbf{x})$ (the potential strength V_1 and the lattice period Λ). Our interest lie in obtaining asymptotic stationary states which can be described as individual or canonical

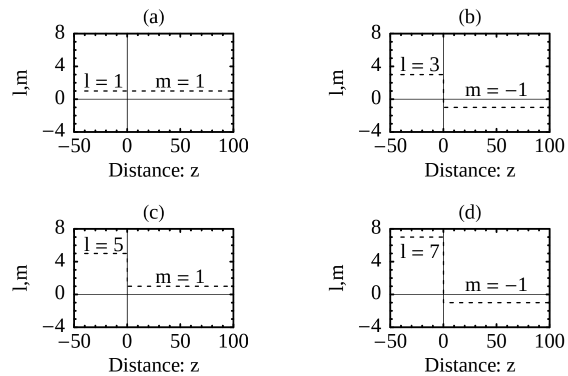

Figure 2: Numerical confirmation of the angular momentum "pass rule" for different values of the angular momentum l of the incident vortex field: (a) $l = 1$ and $m = 1$; (b) $l = 3$ and $m = -1$; (c) $l = 5$ and $m = 1$; (d) $l = 7$ and $m = -1$.

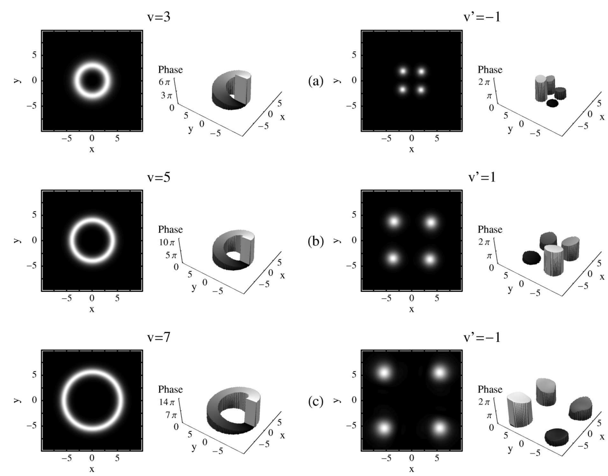

Figure 3: Amplitudes and phases of input $O(2)$ vortices and output $\mathcal{C}_4$ vortices for different values of the input vorticity v: (a) $v = 3$ and $v' = -1$ ($P = 2.5$, $\mu = 2$, $V_1 = 2$, $\Lambda = 1.3$); (b) $v = 5$ and $v' = 1$ ($P = 3.0$, $\mu = 3$, $V_1 = 3$, $\Lambda = 2.5$); (c) $v = 7$ and $v' = -1$ ($P = 3.5$, $\mu = 3$, $V_1 = 3$, $\Lambda = 3.5$).

discrete-symmetry vortices. This condition implies that the asymptotic field has to present a single phase singularity. In other words, we want to exclude multi-vortex or cluster excitations. In order to achieve this feature, we enlarge the optical lattice (by increasing its period Λ) according to the size of the input vortex for increasing values of l. Thus, in our simulations Λ is fixed by l.

After performing many different simulations, we have indeed found numerical evidence of the "vortex transmutation" phenomenon. By playing with the input parameters P and μ and the lattice strength V_1 and period Λ, we have been able to find asymptotic stationary states $\phi_m^{\mathrm{v}} = e^{im\theta} g_m(r, \theta) e^{-i\mu' z}$ for different values of the input vorticity value $v = l$. The vorticity of the output field can be only $v' = \pm 1$ because of the vortic-

ity cutoff for a $\mathcal{C}_4$ system (recall that $m = \pm 2$ solutions are not vortices but nodal or dipole-mode solitons [11]). In Fig. 3 we show the amplitudes and phases of input and output vortices for different input vorticity values v. All of them verify the vorticity "pass rule" (2). The "vortex transmutation" phenomenon only occurs when $|v| > 2$. Similar results are found for the corresponding input anti-vortices with negative values of v. When, for fixed $v = l$ (fixed Λ), the election of P, μ, and V_1 is not adequate the asymptotic solution can be non-stationary. We observe two different scenarios besides the stationary regime: discrete diffraction of the input wave in the optical lattice and self-focussing instability leading to filamentation of the field. A thorough analysis of multiple configurations permits to elaborate a "vortex transmutation" phase diagram where the three different regimes can be recognized. As an example, in the phase diagram shown in Fig. 4 we observe the "vortex transmutation" region (shaded) differentiated from the diffraction (white) and self-focussing instability (light shaded) regions as a function of the power P and propagation constant μ of the input vortex field at fixed V_1. Analogous phase diagrams are found for different values of V_1. It is interesting to analyze the evolution of different "vortex-transmuting" configurations by monitoring the evolution of the power $P'(z)$ and average propagation constant $\mu'(z) \equiv \int \phi^*(-i\partial/\partial z)\phi / \int \phi^*\phi$ (defined both on the finite domain of the numerical solution) in the $\mathcal{C}_4$ medium. These quantities are z-dependent, in general. However, they become independent of z when we analyze a stationary solution; thus we expect $(P'(z), \mu'(z)) \overset{z\to\infty}{\longrightarrow} (P', \mu')$ for asymptotic stationary states. Every input $O(2)$ vortex characterized by the initial values (P, μ) defines then a different trajectory in the P'-μ' plane. In Fig. 4 (inset) we show four different trajectories mapping $O(2)$ $v = 3$ vortices with different (P, μ) initial values into asymptotic $\mathcal{C}_4$ vortices with charge $v' = -1$ characterized by their (P', μ') values. It can be checked numerically that these values lie on the same $P'(\mu')$ curve found in Ref.[14] for stationary vortices with charge $v' = -1$ in an identical square optical lattice. By launching a whole family of initial vortices we have been able to asymptotically reproduce the entire $P'(\mu')$ curve of $\mathcal{C}_4$ vortices. It is remarkable that the asymptotic $\mathcal{C}_4$ vortices in the Fig. 4 inset have been checked to be stable under small perturbations [14] whereas the original $O(2)$ ones are not [1]. Thus the "vortex transmutation" phenomenon not only permits to change the charge of unstable input vortices but it can also help to transform them into stable structures. Inversion of the vortex charge has been observed in "noncanonical" vortices in free-space and occurs through its dynamic propagation [13]. Here, however, all vortices involved are "canonical", the key point to the "vortex transmutation" phenomenon to occur being the suitable matching between angular and pseudo-angular momentum at the $O(2)$-$\mathcal{C}_n$ interface. Despite our model

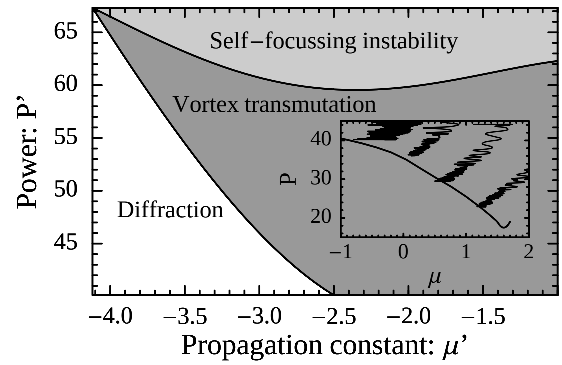

Figure 4: "Vortex transmutation" phase diagram for $v = 3$ vortices in an optical lattice with $V_1 = 2$. Inset: Evolution in the (P', μ') plane of four initial $v = 3$ vortices characterized by different initial (P, μ) values. The solid line curve corresponds to the stationary $v' = -1$ vortices as in Ref.[14].

refers to an specific system, the theory of transmuting vortices is general and applies to any system given by an equation of the type $L(|\phi|)\phi = -i\partial\phi/\partial z$ in the presence of an $O(2)$-$\mathcal{C}_n$ interface. Hence, it is expected this phenomenon to occur in a wide variety of physical systems as those mentioned in the introduction of this Letter.

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